# Semestral

## Number Theory

**Instructor:** Ramdin Mawia  Marks: **50**  Course: M1  **Time:** May 2, 2024; 10:00–13:00.

***Attempt any FIVE among problems 1–6.*** [¶] ***Problem 7 is a bonus problem. Each question, except the bonus problem, carries 10 marks.***

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1. Prove or disprove:

   i. For $n \in \mathbb{Z}$, any odd prime divisor of $n^4+1$ is of the form $8k+1$ for some $k \in \mathbb{N}$. **6**

   ii. The sum of digits of $7^n - 6n - 1$ (in base 10) is divisible by 9 for any $n \in \mathbb{N}$. **4**

2. Let $p$ be an odd prime. Prove or disprove:

   i. The integer $n^{p-1} + (p-1)^n$ is not a prime for any integer $n > 1, n \not\equiv 0 \pmod{p}$. **6**

   ii. If $p \equiv -1 \pmod 4$ then $N^2 \equiv 1 \pmod p$, where $N$ is the product of all even positive integers less than $p$. **4**

3. Describe all primes $p$ for which 10 is a quadratic residue mod $p$. **10**

4. Let $\alpha > 0$, and let $f, g : [1,\infty) \to \mathbb{C}$ be related by $g(x) = \sum_{n \leqslant x^{1/\alpha}} \frac{1}{n^\alpha} f\left(\frac{x}{n^\alpha}\right)$. Prove that $f(x) = \sum_{n \leqslant x^{1/\alpha}} \frac{\mu(n)}{n^\alpha} g\left(\frac{x}{n^\alpha}\right)$ for all $x \geqslant 1$. **2+3+5 =10**

   i. Let $f(x) = \sum_{\substack{(u,v)=1 \\ uv \leqslant x}} 1/uv$ and $g(x) = \sum_{uv \leqslant x} 1/uv$. Show that
   $$g(x) = \sum_{n \leqslant x^{1/2}} \frac{1}{n^2} f\left(\frac{x}{n^2}\right).$$

   ii. Using i. or otherwise, prove that
   $$f(x) = \frac{3}{\pi^2} \log^2 x + O(\log x).$$

5. Find all pairs $(x, y)$ of rational integers satisfying $x^3 = y^2 + 2$. **10**

6. Let $d < 0$ be a squarefree integer and let $A = \mathbb{Z}[\sqrt{d}]$. For integers $a, b$ with $a > 0$, let $I_{a,b}$ denote the additive subgroup $a\mathbb{Z} + (b+\sqrt{d})\mathbb{Z}$ of $A$.

   i. Prove that $I_{a,b}$ is an ideal of $A$ if and only if $b^2 \equiv d \pmod a$. **1**

   ii. Prove that $|A/I_{a,b}| = a$. **2**

   iii. If $I$ is a principal ideal of $A$ prove that $|A/I|$ is either a square or $\geqslant |d|$. [§] **5**

   iv. Conclude that if $A$ is a PID then $d = -1$ or $d = -2$. **2**

[BONUS] 7. Let $p$ be a prime, and let $|\cdot|_p$ denote the $p$-adic absolute value on $\mathbb{Q}$. [‡]

   i. Describe the valuation ring $A_p := \{x \in \mathbb{Q} : |x|_p \leqslant 1\}$. **1**

   ii. Show that $A_p$ is integrally closed. **2**

   iii. Prove that $A_p$ is a local ring, with $A_p/\mathfrak{m}_p \cong \mathbb{Z}/p\mathbb{Z}$, where $\mathfrak{m}_p$ is the unique maximal ideal of $A_p$. **2**

**The End**

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[¶] You may use results proved in class, unless you are asked to prove the result itself.

[§] *Hint.* What is $|A/I|$ when $I$ is principal?

[‡] This is a bonus problem. You may attempt it in addition to five other problems.